Quantum correlations VII

Wladyslaw Adam Majewski

Instytut Fizyki Teoretycznej i Astrofizyki, UG
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland;

- Entanglement of Formation.

- EoF - entanglement of formation is the second mathematical 'tool' for an analysis of entanglement.

- It is designed to separate separable states from entangled states.

- EoF was introduced by Bennett, DiVincenzo, Smolin and Wooters for finite dimensional case in 1996.

- The general definition of EoF for quantum systems (so for infinite dimensional cases) was appeared in 2002.

- EoF can be considered as a measure of entanglement, so as a measure of quantum correlations.

- The basic idea stems from the following observation:

- Let ω be a separable state on a quantum composite system specified by $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$.

- Decompose ω into pure states and apply the restriction map $r_1 : \mathfrak{S}_{\mathfrak{A}} \to \mathfrak{S}_{\mathfrak{A}_1}$, given by $r_1\omega(A) = \omega(A \otimes \mathbb{1})$, to each component of the decomposition.

- Let $\mathbb{F}$ be a function defined on the set of states such that it takes the value 0 only on pure states.

- Then, applying $\mathbb{F}$ to the restriction of each component one gets an indicator of separability.

- Why?

- To answer, it is important to observe that the restriction of a pure state is a pure one only for certain exceptional cases.

- To clarify this question we provide relevant results. The first one is (is extracted from Takesaki's book):

- **Proposition 1.** *Let* $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, *where* $\mathfrak{A}_i$, $i = 1, 2$ *is a* C^**-algebra, and the state* $\omega \in \mathfrak{S}_{\mathfrak{A}}$ *be given. Denote by* $r_1\omega$ *a restriction of state* ω *to* $\mathfrak{A}_1$ *(we identify* $\mathfrak{A}_1$ *with* $\mathfrak{A}_1 \otimes \mathbb{1}_2$*). Assume that* $r_1\omega$ *is a pure state. Then,* $\omega(AB) = \omega(A)\omega(B)$ *when* $A \in \mathfrak{A}_1$ *and* $B \in \mathfrak{A}_2$.

- Thus, the purity of $r_1\omega$ implies the factorization of ω.

- The second result is (again taken from Takesaki's book):

- **Theorem 2.** *For two C*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ the following conditions are equivalent*

  1. *Either $\mathfrak{A}_1$ or $\mathfrak{A}_2$ is abelian*
  2. *Every pure state ω on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is of the form $\omega = \omega_1 \otimes \omega_2$ for some pure states ω_i on $\mathfrak{A}_i$, $i = 1, 2$.*

- Thus, the restriction of a state is a pure state for exceptional cases only!

- In our lectures, we consider a quantum composite system.

- It means that both subsystems are quantum. Consequently, neither $\mathfrak{A}_1$ nor $\mathfrak{A}_2$ is abelian.

- To give the next result some preliminaries are necessary.

- Given a pair $(\mathfrak{A}, \omega)$ consisting of a C^*-algebra and a state, one can associate (via GNS construction) the Hilbert space $\mathcal{H}_\omega$ and the representation π_ω.

- The family of all bounded linear operators on $\mathcal{H}_\omega$, as usually, will be denoted by $B(\mathcal{H}_\omega)$.

- The commutant of $\pi_\omega(\mathfrak{A})$ is defined as $\pi_\omega(\mathfrak{A})' = \{A \in B(\mathcal{H}_\omega); A\pi_\omega(B) = \pi_\omega(B)A \quad \text{for all B} \in \mathfrak{A}\}$.

- In the same way one can define bicommutant $\pi_\omega(\mathfrak{A})'' = (\pi_\omega(\mathfrak{A})')'$.

- $\pi_\omega(\mathfrak{A})''$ is said to be a factor if $\pi_\omega(\mathfrak{A})'' \cap \pi_\omega(\mathfrak{A})' = \{\mathbb{C}\mathbb{1}\}$.

- **Definition 3.** *A state ω on a C*-algebra $\mathfrak{A}$ is said to be factorial if $\pi_\omega(\mathfrak{A})''$ is a factor.*

- The promised result is:

- **Proposition 4.** *Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be C*-algebras and put $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$. Denote by r_1 the restriction map $r_1 : \mathfrak{S}_{\mathfrak{A}} \to \mathfrak{S}_{\mathfrak{A}_1}$. r_1 is weak-$*$ continuous. Moreover, $Ext(\mathfrak{S}_{\mathfrak{A}_1}) \subseteq r_1(Ext(\mathfrak{S}_{\mathfrak{A}})) \subseteq F_{\mathfrak{A}_1}$ where $F_{\mathfrak{A}_1}$ stands for factorial states on $\mathfrak{A}_1$, and $Ext(\mathfrak{S}_{\mathfrak{A}})$ stands for the subset of extreme elements of $\mathfrak{S}_{\mathfrak{A}}$. If $\mathfrak{A}_2$ is abelian then $Ext(\mathfrak{S}_{\mathfrak{A}_1}) = r_1(Ext(\mathfrak{S}_{\mathfrak{A}}))$.*

- Consequently, if one considers the true quantum composite system, i.e. both subsystems are quantum (so both C^*-algebras $\mathfrak{A}_i$ are non commutative ones) then one can say only that the restriction of a pure state is a factorial one.

- This clearly indicates the role of the function $\mathbb{F}$ in the definition given below.

- Also, it is worth pointing out that (weak-*)-(weak-*) continuity of the restriction r_1 was already used in the discussion of quantum coefficient of correlations.

- We give:

Definition 5. *Let ω be a state, $\omega \in F \subset \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ and F satisfy separability condition SC. The entanglement of formation EoF is defined as*

$$E_{\mathbb{F}}(\omega) = \inf_{\mu \in M_\omega(\mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2})} \int \mathbb{F}(r\varphi) d\mu(\varphi) \tag{1}$$

where $\mathbb{F}$ is a concave non-negative continuous function which vanishes on pure states and only on pure states, and to shorten notation we write r instead of r_1.

- Let us comment upon this definition.

- Firstly, we recall that a given state ω can have many decompositions.

- Therefore, we are forced to use the decomposition theory.

- In particular, orthogonal measures are playing an important role as they could be supported by $Ext(\mathfrak{S}_{\mathfrak{A}})$!

- Further, we assumed the separability condition, SC, to avoid pathological measure-theoretical cases in the decomposition of ω.

- But, Ruelle's SC condition holds for all essential physical models.

- Finally, in physical literature, frequently, one employs the von Neumann entropy as the function $\mathbb{F}$.

- **Example 6.** *Let* $\mathfrak{A}_i = B(\mathcal{H}_i)$, $i = 1, 2$; $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ *and* F *denote the set of all normal states on* $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

  Let $\varphi \in F$. *Then* $\varphi(\cdot) = Tr\rho_\varphi(\cdot)$, *and one can identify* φ *with* ρ_φ. *Take* $\mathbb{F}$ *to be the von Neumann entropy*

$$\mathbb{F}(\varphi) = -\mathrm{Tr}\rho_\varphi \log \rho_\varphi \tag{2}$$

  Clearly, the von Neumann entropy satisfies all necessary properties provided that $\mathcal{H}$ *is finite dimensional.*

  However, note that the function $\rho_\varphi \mapsto \mathrm{Tr}\{\rho_\varphi(\mathbb{1} - \rho_\varphi)\}$ *also possesses all necessary properties (again, for finite dimensional systems).*

  Thus, this function can lead to another measure of entanglement.

- We wish to show that Entanglement of Formation can distinguish separable states from entangled states.

- To simplify notation, in the sequel, we will write $E(\omega)$ instead of $E_{\mathbb{F}}(\omega)$.

- $E(\omega)$ is defined as infimum of integrals evaluated on a continuous functions and the infimum is taken over the compact set.

- Therefore, the infimum is attainable, i.e. there exists a measure $\mu_0 \in M_\omega(\mathfrak{S})$ such that

$$E(\omega) = \int_{\mathfrak{S}} \mathbb{F}(r\varphi) d\mu_0(\varphi) \tag{3}$$

- and, obviously,

$$\int_{\mathfrak{S}} \varphi d\mu_0(\varphi) = \omega \tag{4}$$

- Let us assume that $E(\omega) = 0$.

- Then we have

$$\int_{\mathfrak{S}} \mathbb{F}(r\varphi) d\mu_0(\varphi) = 0 \tag{5}$$

- As $\mathbb{F}(r\varphi)$ is non-negative, one can infer that $\mathbb{F}(r\varphi) = 0$ on the support of μ_0.

- But, as $\mathbb{F}$ is a concave function, one has

$$\mathbb{F}(r\varphi) \geq \int_{\mathfrak{S}} \mathbb{F}(rv) d\xi(v) \tag{6}$$

for any probability measure $d\xi$ on $\mathfrak{S}$ such that $\varphi = \int_{\mathfrak{S}} v d\xi(v)$.

- In particular taking (as a measure ξ) a measure supported by pure states (from decomposition theory such measures exist) we conclude the existence of decomposition of φ such that $\mathbb{F}(rv) = 0$ for v, hence rv is a pure state and consequently v is a product state, cf Proposition 1.

- So, φ is a convex combination of product states.

- Due to the fact that any (classical) measure has weak*-approximation property, ω can be approximated by a convex combination of product states.

- Consequently, ω is a separable state.

Conversely, let ω be a separable state, i.e.

$$\omega = \lim_{N \to \infty} \sum_{i=1}^{N} \lambda_i^{(N)} \omega_i^{(N)} \tag{7}$$

where each ω_i is a product state such that $\omega_i^{(N)}(A \otimes B) = \omega_{i,1}^{(N)}(A)\omega_{i,2}^{(N)}(B)$,

where $\omega_{i,k}^{(N)}(\cdot)$ is a pure state on $\mathfrak{A}_k$.

- Define the sequence of measures $\mu^{(N)}$ in the following way:

$$\mu^{(N)} = \sum_{i=1}^{N} \lambda_i^{(N)} \delta_{\omega_i^{(N)}} \tag{8}$$

where $\delta_{\omega_i^{(N)}}$ denotes Dirac's measure.

- If necessary, passing to a subsequence, we may suppose also that $\mu^{(N)}$ converges to $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$

  (it is always possible as $\{\mu^{(N)}\} \subset M_1(\mathfrak{S}_\mathfrak{A})$, which a compact set).

- Taking a weak limit of $\left\{\mu^{(N)}\right\}$ one gets a measure μ such that

$$\int \varphi d\mu(\varphi) = \omega \tag{9}$$

and

$$\int \mathbb{F}(r\varphi) d\mu(\varphi) = 0 \tag{10}$$

- Thus, we arrived at:

  Theorem 7. $E(\omega) = 0$ *if and only if* $\omega \in F$ *is separable.*

- Entanglement of Formation, EoF, is not only a nice indicator of separability.

- It possesses also many useful properties like convexity, semi-continuity and others.

- In this lecture, we will be concerned with convexity and with the property of EoF which can be regarded as an analogue of entanglement witness.

- We begin with

Proposition 8. $\mathfrak{S}_{\mathfrak{A}} \ni \omega \longrightarrow E(\omega)$ *is a convex function.*

- To see this we note that decomposition theory implies

- $\mu \in M_\omega(\mathfrak{S}_{\mathfrak{A}})$ if and only if $\mu(f) \geq f(\omega)$ for any real, continuous convex function on $\mathfrak{S}_{\mathfrak{A}}$.

- Thus,

if $\mu_1 \in M_{\omega_1}(\mathfrak{S}_{\mathfrak{A}})$, $\mu_2 \in M_{\omega_2}(\mathfrak{S}_{\mathfrak{A}})$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\lambda_1 + \lambda_2 = 1$

then

$$(\lambda_1\mu_1+\lambda_2\mu_2)(f) = \lambda_1\mu_1(f)+\lambda_2\mu_2(f) \geq \lambda_1 f(\omega_1)+\lambda_2 f(\omega_2) \geq f(\lambda_1\omega_1+\lambda_2\omega_2).$$

- On the other hand

$$\lambda_1\mu_1 + \lambda_2\mu_2 \in M_{\lambda_1\omega_1+\lambda_2\omega_2}(\mathfrak{S}_{\mathfrak{A}})$$

is equivalent to

$$(\lambda_1\mu_1 + \lambda_2\mu_2)(f) \geq f(\lambda_1\omega_1 + \lambda_2\omega_2).$$

- But, the convexity of f implies $\lambda_1 f(\omega_1) + \lambda_2 f(\omega_2) \geq f(\lambda_1\omega_1 + \lambda_2\omega_2)$.

- Therefore, $\lambda_1\mu_1 + \lambda_2\mu_2 \in M_{\lambda_1\omega_1+\lambda_2\omega_2}(\mathfrak{S}_{\mathfrak{A}})$.

- Consequently

$$\lambda_1 M_{\omega_1}(\mathfrak{S}_{\mathfrak{A}}) + \lambda_2 M_{\omega_2}(\mathfrak{S}_{\mathfrak{A}}) \subseteq M_{\lambda_1\omega_1+\lambda_2\omega_2}(\mathfrak{S}_{\mathfrak{A}}). \tag{11}$$

- Hence

$$E(\lambda_1\omega_1 + \lambda_2\omega_2) = \inf_{\mu \in M_{\lambda_1\omega_1+\lambda_2\omega_2}(\mathfrak{S}_{\mathfrak{A}})} \int \mathbb{F} \circ r(\varphi) d\mu(\varphi) \qquad (12)$$

$$\leq \lambda_1 \inf_{\mu \in M_{\omega_1}(\mathfrak{S}_{\mathfrak{A}})} \int \mathbb{F} \circ (\varphi) d\mu(\varphi) + \lambda_2 \inf_{\mu \in M_{\omega_2}(\mathfrak{S}_{\mathfrak{A}})} \int \mathbb{F} \circ (\varphi) d\mu(\varphi)$$
$$= \lambda_1 E(\omega_1) + \lambda_2 E(\omega_2).$$

- Thus, EoF is a convex function.

- EoF has a property which seems to be of the same nature as entanglement witness.

- To describe it we need some preliminaries.

- The first one is Bauer maximum principle:

Proposition 9. *Let E be a Hausdorff locally convex topological space and $X \subset E$ a (non empty) convex compact subset. Suppose $f : X \longrightarrow \mathbb{R}$ is convex and upper semi-continuous. Then there exists an extreme point of X (not necessarily unique) at which f assumes its maximum value.*

- The second one is the characterization of extremal measures in $M_\omega(\mathfrak{S}_\mathfrak{A})$.

Proposition 10. *Let $\mathfrak{A}$ be a C^*-algebra (with identity) and $\omega \in \mathfrak{S}_\mathfrak{A}$. Let μ be in $M_\omega(\mathfrak{S}_\mathfrak{A})$. Then, the following conditions are equivalent:*

1. $\mu \in Ext\,(M_\omega(\mathfrak{S}_\mathfrak{A}))$.
2. *the affine continuous functions over $\mathfrak{S}_\mathfrak{A}$ are dense in $L^1(\mathfrak{S}_\mathfrak{A}, \mu)$.*

- "new entanglement witness".

- To describe it, we denote, for $\mu \in M_\omega(\mathfrak{S}_\mathfrak{A})$

$$E_\mu(\omega) = \int (\mathbb{F} \circ r)(\varphi) d\mu(\varphi).$$

- $M_\omega(\mathfrak{S}_\mathfrak{A}) \ni \mu \mapsto E_\mu(\omega)$ is an affine, real valued, continuous function on $\mathfrak{S}_\mathfrak{A}$.

- An application of Proposition 9 implies that minimum in definition of $E(\omega)$ (cf the proof of Theorem 7) is attained on a certain extremal measure μ_0 in $M_\omega(\mathfrak{S}_\mathfrak{A})$.

- Then, applying Proposition 10 one gets:

- there exists an affine function $\hat{A}_0 : \omega \mapsto \omega(A_0)$, where $A_0^* = A_0 \in \mathfrak{A}$ such that

$$|E(\omega) - \omega(A_0)| = |\int_{\mathfrak{S}_{\mathfrak{A}}} (\mathbb{F} \circ r)(\varphi) d\mu_0(\varphi) - \int_{\mathfrak{S}_{\mathfrak{A}}} \hat{A}_0(\varphi) d\mu_0(\varphi)| \quad (13)$$

$$\leq \int_{\mathfrak{S}_{\mathfrak{A}}} |(\mathbb{F} \circ r)(\varphi) - \hat{A}_0(\varphi)| d\mu_0(\varphi) < \epsilon,$$

for an arbitrary small ϵ, as $\mathbb{F} \circ r$ is a continuous function on a compact set $\mathfrak{S}_{\mathfrak{A}}$.

- Consequently, there exists an observable $A_0 = A_0^* \in \mathfrak{A}$ such that its expectation value $\omega(A_0) \equiv < A_0 >_\omega$ approximates EoF, $E(\omega)$, at a given state ω.

- Final remark.

- continuity of EoF, i.e. of the mapping $\mathfrak{S}_{\mathfrak{A}} \ni \omega \mapsto E(\omega)$.

- This mapping is a real valued, convex function defined on a compact set.

- It can be proved that it is lower semicontinuous.

- In general, not upper semicontinuos.

- Consequently, continuity properties of EoF, $E(\omega)$ are of the same sort as those of quantum entropy, see Wehrl.